Pair counting — Up to 25 members

Name: ____________________ Date: __________

Count the pairs without listing them. Show your working.

1 A league has 9 teams. Every team plays every other team once.

How many matches are played?

Why Each of the 9 meets the other 8, which is $9 \times 8 = 72$. But that counts every pair twice — once from each end — so halve it: $72 \div 2 = 36$ matches. More than one way works here, and the other needs no halving at all: the first meets 8, the second meets 7 it has not met yet, and so on down to 1. Adding $8 + 7 + \ldots + 1$ gives the same 36. This method counts every pair once from the start, so there is nothing to halve at the end.

36

2 25 offices are connected so that each one can call each of the others directly, and a call from A to B is not the same as a call from B to A.

How many different calls are possible?

Why

______

Name: ____________________ Date: __________

Count the pairs without listing them. Show your working.

3 12 friends sit down to eat, and every pair of them clinks glasses once.

How many clinks are there?

Why

___

4 14 offices are connected so that each one can call each of the others directly, and a call from A to B is not the same as a call from B to A.

How many different calls are possible?

Why

____

Diagonal counting — Up to 20 corners

Name: ______________________ Date: ____________

Count them without drawing them. Show your working.

1. 5 towns are arranged in a ring, and a road already joins each town to the two beside it. New roads are built so that every pair of towns is joined directly.

How many new roads are built?

Why Every pair of the 5 could be joined. Each town reaches 4 others, so 5 × 4 = 20 — and each pair is counted from both ends, so 20 ÷ 2 = 10 lines in all. But 5 of those roads already exist, so take them off: 10 – 5 = 5 new roads. More than one way works here and it never counts the roads already there: each town joins all but itself and its two neighbors, so 5 × 2 = 10, and 10 ÷ 2 = 5.

5

2. 5 people sit in a ring. Everyone shakes hands with everyone else except the two sitting next to them.

How many handshakes are there?

Why

___

Diagonal counting — Up to 20 corners

Name: ______________________ Date: ______________

Count them without drawing them. Show your working.

3 A polygon has 7 sides.

How many diagonals does it have?

Why

___

4 17 people sit in a ring. Everyone shakes hands with everyone else except the two sitting next to them.

How many handshakes are there?

Why

___

Lattice paths — Grids up to 10

Name: ______________________ Date: ____________

Count the routes across a grid, moving only right or up, and show why it is a combination, not a power of two.

1 An ant walks a park 4 blocks wide and 3 blocks tall, from one corner to the opposite one, only ever going right or up one block at a time.

How many different routes are there?

Why Every route is 7 steps — 4 to the right and 3 up — so a route is just which 3 of the 7 steps go up. That is 7 choose 3, built up one factor at a time: 1 × 7 = 7, 7 ÷ 1 = 7, 7 × 6 = 42, 42 ÷ 2 = 21, 21 × 5 = 105, 105 ÷ 3 = 35. So 35 routes. It is not 128 (two choices at every step would count routes that run off the grid).

35

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2 An ant walks a park 5 blocks wide and 2 blocks tall, from one corner to the opposite one, only ever going right or up one block at a time.

How many different routes are there?

Why

Lattice paths — Grids up to 10

Name: ______________________ Date: __________

Count the routes across a grid, moving only right or up, and show why it is a combination, not a power of two.

3 A city grid is 2 blocks wide and 6 blocks tall. Starting at the bottom-left corner and walking only right or up along the streets, count the different routes to the top-right corner.

How many different routes are there?

Why

4 An ant walks a park 3 blocks wide and 5 blocks tall, from one corner to the opposite one, only ever going right or up one block at a time.

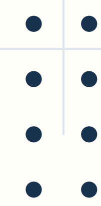
How many different routes are there?

Why

Name: ______________________________ Date: ______________

Count the triangles you can make from a grid of dots, and show why the straight lines come off.

1 An orchard is planted in a rectangle of 2 trees by 4 trees, 8 trees in all. A triangle is any three trees taken as its corners.



How many triangles can be made?

Why Every triangle is just 3 of the 8 trees chosen as corners, so start from 8 choose 3, C(8,3) = 56, built one factor at a time: $1 \times 8 = 8$, $8 \div 1 = 8$, $8 \times 7 = 56$, $56 \div 2 = 28$, $28 \times 6 = 168$, $168 \div 3 = 56$. But three trees on one straight line make no triangle. The straight-line triples here are 8 down the columns — 8 in all. Take them off: $56 - 8 = 48$. So 48 triangles, not the 56 you get by forgetting that some threes are in a line.

48

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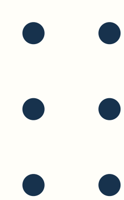
2 A geoboard has its pegs in a rectangle 2 pegs across and 3 pegs down — 6 pegs altogether. A triangle is any three pegs joined by rubber bands.



How many triangles can be made?

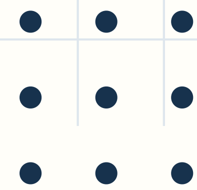
Why

Counting triangles — Grids up to 12 dots

Name: ______________________ Date: ____________

Count the triangles you can make from a grid of dots, and show why the straight lines come off.

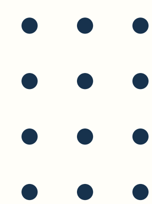
3 A grid of dots is 3 across and 3 down, so 9 dots in all. A triangle is any three of the dots joined up as corners.



How many triangles can be made?

Why

4 An orchard is planted in a rectangle of 3 trees by 4 trees, 12 trees in all. A triangle is any three trees taken as its corners.



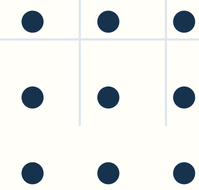
How many triangles can be made?

Why

Name: ____________________ Date: __________

Count the rectangles you can make from a grid of dots, and show why it is a product of two combinations.

1 A pegboard holds its pegs in a rectangle 3 pegs across and 3 pegs down — 9 pegs altogether. A rectangle is any four pegs that make its corners, sides along the rows and columns.



How many rectangles are there?

Why A rectangle with sides on the grid is pinned down by 2 of the 3 columns — its left and right edges — and 2 of the 3 rows — its top and bottom. The columns give C(3,2) = 3, the rows give C(3,2) = 3, and every pair of columns goes with every pair of rows: 3 × 3 = 9. So 9 rectangles, not the 4 little cells you get by counting only the smallest squares.

9

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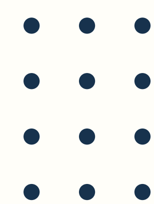
2 A pegboard holds its pegs in a rectangle 3 pegs across and 4 pegs down — 12 pegs altogether. A rectangle is any four pegs that make its corners, sides along the rows and columns.



How many rectangles are there?

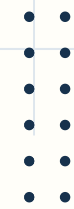
Why

____

Name: ______________________ Date: ______________

Count the rectangles you can make from a grid of dots, and show why it is a product of two combinations.

3 A park has lamp posts in a grid 2 across and 6 deep, 12 lamps in all. A rectangle has four lamps as corners with its sides running along the rows and columns.



How many rectangles are there?

Why

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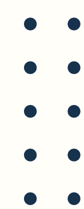
4 A grid of dots is 2 across and 5 down, so 10 dots in all. A rectangle has four of the dots as corners and its sides run along the grid.



How many rectangles are there?

Why

Lazy caterer — Up to 10 cuts

Name: ______________________ Date: __________

A pancake is cut by straight lines. Find the most pieces, and show why it stops doubling.

1. A pizza is cut with 9 straight cuts, each going all the way across, and nothing is rearranged between cuts.

What is the greatest number of pieces?

Why The most pieces come from 1 whole pizza, plus one for each cut and one for every PAIR of cuts (a pair crosses once inside). The cuts are C(9,1) = 9; the pairs C(9,2) = 36. Adding them, 9 + 36 = 45, and the whole pizza makes 45 + 1 = 46. The doubling 2, 4 would say 512 — but from the third cut on, it is wrong.

46

2. A pie is sliced 6 times, every slice a straight line across the whole pie, with no piece moved in between.

What is the greatest number of pieces?

Why

Name: ______________________ Date: ____________

A pancake is cut by straight lines. Find the most pieces, and show why it stops doubling.

3 A round pancake is cut by 5 straight cuts, each cut a full line right across it, and no piece is moved between cuts.

What is the greatest number of pieces?

Why

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4 A pizza is cut with 2 straight cuts, each going all the way across, and nothing is rearranged between cuts.

What is the greatest number of pieces?

Why

Power Pack · Counting puzzles — Answer key

If they are stuck

- Pick one of them and count who they meet. Is that true of every one of them?
- Join every pair first, including the ones already joined. What do you then take off?
- Do not draw the routes. Every route is the same length; count which of its steps go right, and that is a combination.
- A triangle is a choice of three dots. Count all the choices, then take off the threes that sit in a straight line.
- A rectangle is two columns and two rows. Count the ways to choose each, and multiply — do not just count the little cells.
- Doubling stops working. Count the cuts, then the pairs of cuts that cross, then add one.

Counting pairs

1. Each of the 9 meets the other 8, which is 9 × 8 = 72. But that counts every pair twice — once from each end — so halve it: 72 ÷ 2 = 36 matches. More than one way works here, and the other needs no halving at all: the first meets 8, the second meets 7 it has not met yet, and so on down to 1. Adding 8 + 7 + ... + 1 gives the same 36. This method counts every pair once from the start, so there is nothing to halve at the end.

2. Each of the 25 meets the other 24, and this time the two directions are different events. So 25 × 24 = 600 calls, with no halving.

3. Each of the 12 meets the other 11, which is 12 × 11 = 132. But that counts every pair twice — once from each end — so halve it: 132 ÷ 2 = 66 clinks. More than one way works here, and the other needs no halving at all: the first meets 11, the second meets 10 it has not met yet, and so on down to 1. Adding 11 + 10 + ... + 1 gives the same 66. This method counts every pair once from the start, so there is nothing to halve at the end.

4. Each of the 14 meets the other 13, and this time the two directions are different events. So 14 × 13 = 182 calls, with no halving.

Power Pack · Counting puzzles — Answer key

If they are stuck

- Pick one of them and count who they meet. Is that true of every one of them?
- Join every pair first, including the ones already joined. What do you then take off?
- Do not draw the routes. Every route is the same length; count which of its steps go right, and that is a combination.
- A triangle is a choice of three dots. Count all the choices, then take off the threes that sit in a straight line.
- A rectangle is two columns and two rows. Count the ways to choose each, and multiply — do not just count the little cells.
- Doubling stops working. Count the cuts, then the pairs of cuts that cross, then add one.

Diagonals without drawing

1. Every pair of the 5 could be joined. Each town reaches 4 others, so $5 \times 4 = 20$ — and each pair is counted from both ends, so $20 \div 2 = 10$ lines in all. But 5 of those roads already exist, so take them off: $10 - 5 = 5$ new roads. More than one way works here and it never counts the roads already there: each town joins all but itself and its two neighbors, so $5 \times 2 = 10$, and $10 \div 2 = 5$.

2. Every pair of the 5 could be joined. Each person reaches 4 others, so $5 \times 4 = 20$ — and each pair is counted from both ends, so $20 \div 2 = 10$ lines in all. But 5 of them join neighbors, so take them off: $10 - 5 = 5$ handshakes. More than one way works here and it never counts the neighbor pairs: each person joins all but itself and its two neighbors, so $5 \times 2 = 10$, and $10 \div 2 = 5$.

3. Every pair of the 7 could be joined. Each corner reaches 6 others, so $7 \times 6 = 42$ — and each pair is counted from both ends, so $42 \div 2 = 21$ lines in all. But 7 of them are its sides, so take them off: $21 - 7 = 14$ diagonals. More than one way works here and it never counts the sides: each corner joins all but itself and its two neighbors, so $7 \times 4 = 28$, and $28 \div 2 = 14$.

4. Every pair of the 17 could be joined. Each person reaches 16 others, so $17 \times 16 = 272$ — and each pair is counted from both ends, so $272 \div 2 = 136$ lines in all. But 17 of them join neighbors, so take them off: $136 - 17 = 119$ handshakes. More than one way works here and it never counts the neighbor pairs: each person joins all but itself and its two neighbors, so $17 \times 14 = 238$, and $238 \div 2 = 119$.

Lattice paths

1. Every route is 7 steps — 4 to the right and 3 up — so a route is just which 3 of the 7 steps go up. That is 7 choose 3, built up one factor at a time: $1 \times 7 = 7$, $7 \div 1 = 7$, $7 \times 6 = 42$, $42 \div 2 = 21$, $21 \times 5 = 105$, $105 \div 3 = 35$. So 35 routes. It is not 128 (two choices at every step would count routes that run off the grid).

2. Every route is 7 steps — 5 to the right and 2 up — so a route is just which 2 of the 7 steps go up. That is 7 choose 2, built up one factor at a time: $1 \times 7 = 7$, $7 \div 1 = 7$, $7 \times 6 = 42$, $42 \div 2 = 21$. So 21 routes. It is not 128 (two choices at every step would count routes that run off the grid).

3. Every route is 8 steps — 2 to the right and 6 up — so a route is just which 2 of the 8 steps go right. That is 8 choose 2, built up one factor at a time: $1 \times 8 = 8$, $8 \div 1 = 8$, $8 \times 7 = 56$, $56 \div 2 = 28$. So 28 routes. It is not 256 (two choices at every step would count routes that run off the grid).

4. Every route is 8 steps — 3 to the right and 5 up — so a route is just which 3 of the 8 steps go right. That is 8 choose 3, built up one factor at a time: $1 \times 8 = 8$, $8 \div 1 = 8$, $8 \times 7 = 56$, $56 \div 2 = 28$, $28 \times 6 = 168$, $168 \div 3 = 56$. So 56 routes. It is not 256 (two choices at every step would count routes that run off the grid).

Power Pack · Counting puzzles — Answer key

If they are stuck

- Pick one of them and count who they meet. Is that true of every one of them?
- Join every pair first, including the ones already joined. What do you then take off?
- Do not draw the routes. Every route is the same length; count which of its steps go right, and that is a combination.
- A triangle is a choice of three dots. Count all the choices, then take off the threes that sit in a straight line.
- A rectangle is two columns and two rows. Count the ways to choose each, and multiply — do not just count the little cells.
- Doubling stops working. Count the cuts, then the pairs of cuts that cross, then add one.

Counting triangles

1. Every triangle is just 3 of the 8 trees chosen as corners, so start from 8 choose 3, C(8,3) = 56, built one factor at a time: 1 × 8 = 8, 8 ÷ 1 = 8, 8 × 7 = 56, 56 ÷ 2 = 28, 28 × 6 = 168, 168 ÷ 3 = 56. But three trees on one straight line make no triangle. The straight-line triples here are 8 down the columns — 8 in all. Take them off: 56 − 8 = 48. So 48 triangles, not the 56 you get by forgetting that some threes are in a line.

2. Every triangle is just 3 of the 6 pegs chosen as corners, so start from 6 choose 3, C(6,3) = 20, built one factor at a time: 1 × 6 = 6, 6 ÷ 1 = 6, 6 × 5 = 30, 30 ÷ 2 = 15, 15 × 4 = 60, 60 ÷ 3 = 20. But three pegs on one straight line make no triangle. The straight-line triples here are 2 down the columns — 2 in all. Take them off: 20 − 2 = 18. So 18 triangles, not the 20 you get by forgetting that some threes are in a line.

3. Every triangle is just 3 of the 9 dots chosen as corners, so start from 9 choose 3, C(9,3) = 84, built one factor at a time: 1 × 9 = 9, 9 ÷ 1 = 9, 9 × 8 = 72, 72 ÷ 2 = 36, 36 × 7 = 252, 252 ÷ 3 = 84. But three dots on one straight line make no triangle. The straight-line triples here are 3 along the rows, 3 down the columns, 2 on slanted lines — 8 in all. Take them off: 84 − 8 = 76. So 76 triangles, not the 84 you get by forgetting that some threes are in a line.

4. Every triangle is just 3 of the 12 trees chosen as corners, so start from 12 choose 3, C(12,3) = 220, built one factor at a time: 1 × 12 = 12, 12 ÷ 1 = 12, 12 × 11 = 132, 132 ÷ 2 = 66, 66 × 10 = 660, 660 ÷ 3 = 220. But three trees on one straight line make no triangle. The straight-line triples here are 4 along the rows, 12 down the columns, 4 on slanted lines — 20 in all. Take them off: 220 − 20 = 200. So 200 triangles, not the 220 you get by forgetting that some threes are in a line.

Power Pack · Counting puzzles — Answer key

If they are stuck

- Pick one of them and count who they meet. Is that true of every one of them?
- Join every pair first, including the ones already joined. What do you then take off?
- Do not draw the routes. Every route is the same length; count which of its steps go right, and that is a combination.
- A triangle is a choice of three dots. Count all the choices, then take off the threes that sit in a straight line.
- A rectangle is two columns and two rows. Count the ways to choose each, and multiply — do not just count the little cells.
- Doubling stops working. Count the cuts, then the pairs of cuts that cross, then add one.

Counting rectangles

1. A rectangle with sides on the grid is pinned down by 2 of the 3 columns — its left and right edges — and 2 of the 3 rows — its top and bottom. The columns give C(3,2) = 3, the rows give C(3,2) = 3, and every pair of columns goes with every pair of rows: 3 × 3 = 9. So 9 rectangles, not the 4 little cells you get by counting only the smallest squares.

2. A rectangle with sides on the grid is pinned down by 2 of the 3 columns — its left and right edges — and 2 of the 4 rows — its top and bottom. The columns give C(3,2) = 3, the rows give C(4,2) = 6, and every pair of columns goes with every pair of rows: 3 × 6 = 18. So 18 rectangles, not the 6 little cells you get by counting only the smallest squares.

3. A rectangle with sides on the grid is pinned down by 2 of the 2 columns — its left and right edges — and 2 of the 6 rows — its top and bottom. The columns give C(2,2) = 1, the rows give C(6,2) = 15, and every pair of columns goes with every pair of rows: 1 × 15 = 15. So 15 rectangles, not the 5 little cells you get by counting only the smallest squares.

4. A rectangle with sides on the grid is pinned down by 2 of the 2 columns — its left and right edges — and 2 of the 5 rows — its top and bottom. The columns give C(2,2) = 1, the rows give C(5,2) = 10, and every pair of columns goes with every pair of rows: 1 × 10 = 10. So 10 rectangles, not the 4 little cells you get by counting only the smallest squares.

The lazy caterer

1. The most pieces come from 1 whole pizza, plus one for each cut and one for every PAIR of cuts (a pair crosses once inside). The cuts are C(9,1) = 9; the pairs C(9,2) = 36. Adding them, 9 + 36 = 45, and the whole pizza makes 45 + 1 = 46. The doubling 2, 4 would say 512 — but from the third cut on, it is wrong.

2. The most pieces come from 1 whole pie, plus one for each cut and one for every PAIR of cuts (a pair crosses once inside). The cuts are C(6,1) = 6; the pairs C(6,2) = 15. Adding them, 6 + 15 = 21, and the whole pie makes 21 + 1 = 22. The doubling 2, 4 would say 64 — but from the third cut on, it is wrong.

3. The most pieces come from 1 whole pancake, plus one for each cut and one for every PAIR of cuts (a pair crosses once inside). The cuts are C(5,1) = 5; the pairs C(5,2) = 10. Adding them, 5 + 10 = 15, and the whole pancake makes 15 + 1 = 16. The doubling 2, 4 would say 32 — but from the third cut on, it is wrong.

4. The most pieces come from 1 whole pizza, plus one for each cut and one for every PAIR of cuts (a pair crosses once inside). The cuts are C(2,1) = 2; the pairs C(2,2) = 1. Adding them, 2 + 1 = 3, and the whole pizza makes 3 + 1 = 4. The doubling 2, 4 would say 4 — but from the third cut on, it is wrong.