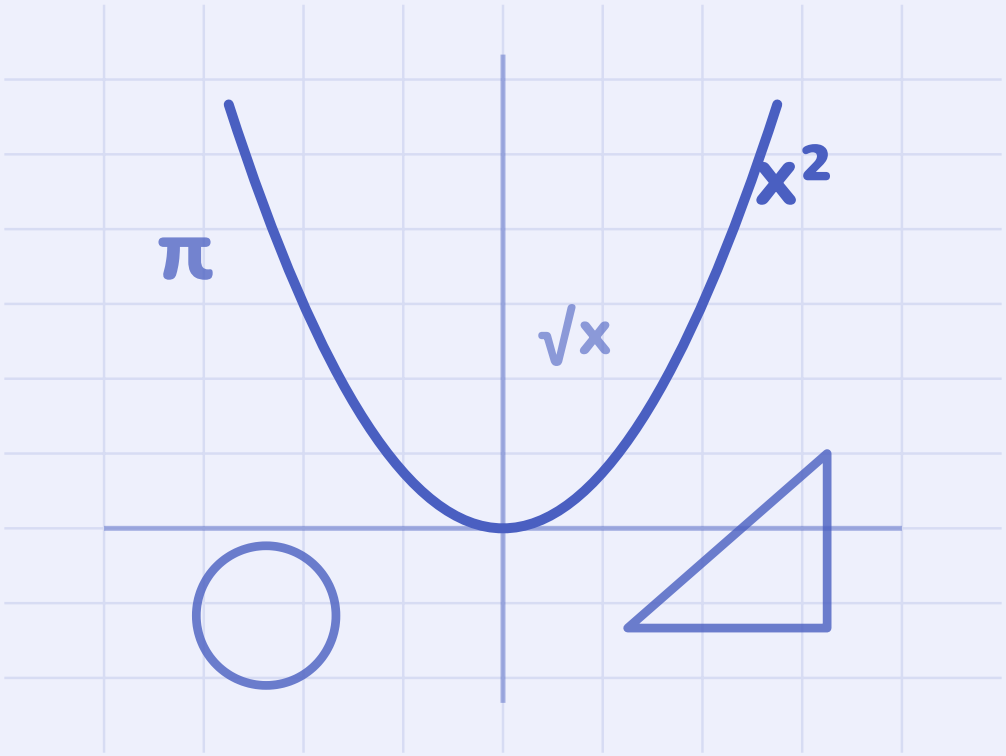
MATH PICNIC · WORKBOOK

Calculus Integration Workbook

5 lessons · Ages 17–18 · answer key at the back



Name _____ Date _____

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Indefinite Integrals

WORKED EXAMPLE

$$\int (4x - 2)\,dx$$
$$= \frac{4x^2}{2} - 2x + C$$
$$= 2x^2 - 2x + C$$

1. Integrals

Raise each power by one and divide by the new power. Don't forget + C.

1 $\int (-4x)\,dx$

2 $\int (2x - 1)\,dx$

3 $\int (-4x + 2)\,dx$

4 $\int (6x)\,dx$

5 $\int (-4x - 1)\,dx$

6 $\int (2x + 3)\,dx$

7 $\int (-2x - 2)\,dx$

8 $\int (4x)\,dx$

9 $\int (-6x - 2)\,dx$

10 $\int (4x - 1)\,dx$

11 $\int (-4x - 2)\,dx$

12 $\int (-4x + 1)\,dx$

2. Derivatives

Find f'(x). Bring each power down in front, and take one off the power.

13 $f(x) = -2x^2 - 3$

14 $f(x) = x^2 - x - 2$

15 $f(x) = -3x^2 + x - 2$

16 $f(x) = -x^2 - 2$

17 $f(x) = -2x^2 + 1$

18 $f(x) = -3x^2 + x + 2$

Definite Integrals

WORKED EXAMPLE

$$\int_0^1 (-4x)\,dx$$
$$= [-2x^2]_0^1$$
$$= (-2(1)^2) - 0$$
$$= -2 - 0$$
$$= -2$$

3. Integrals

Find the antiderivative, then take its value at the bottom from the top.

1 $\int_{-1}^{0} (4x)\,dx$

2 $\int_{-1}^{0} (2x - 2)\,dx$

3 $\int_{0}^{2} (4x)\,dx$

4 $\int_{0}^{1} (4x)\,dx$

5 $\int_{-2}^{2} (4x - 2)\,dx$

6 $\int_{0}^{2} (-4x - 2)\,dx$

7 $\int_{-1}^{1} (-4x + 2)\,dx$

8 $\int_{0}^{1} (4x + 2)\,dx$

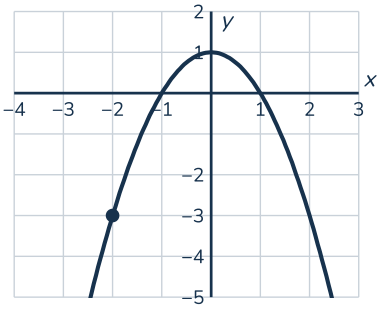
9 $\int_{-1}^{0} (4x + 2)\,dx$

10 $\int_{0}^{2} (-2x)\,dx$

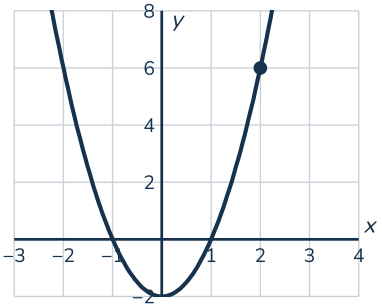
4. Using derivatives

Find the slope of each curve at the point: find f'(x), then put x in.

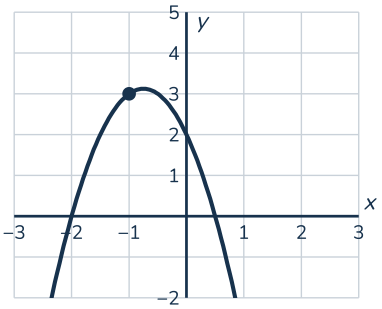
11 **$f(x) = -x^2 + 1,\ x = -2$**



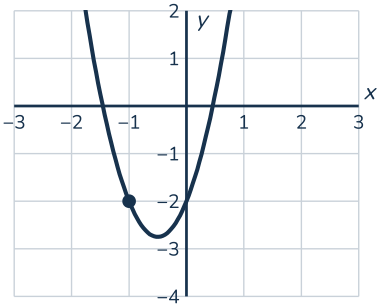
12 **$f(x) = 2x^2 - 2,\ x = 2$**



13 **$f(x) = -2x^2 - 3x + 2,\ x = -1$**



14 **$f(x) = 3x^2 + 3x - 2,\ x = -1$**



Integration by Substitution

WORKED EXAMPLE

$\int -3(-x+3)^2\,dx$

$u = -x + 3,\ du = -\,dx$

$= \int 3u^2\,du$

$= u^3 + C$

$= (-x+3)^3 + C$

5. Integrals

Find each integral by substitution: let u be the bracket.

1 $\int 6(2x-2)^2\,dx$

2 $\int -9(-3x+3)^2\,dx$

3 $\int 6(2x-3)^2\,dx$

4 $\int -12(-3x-3)^3\,dx$

5 $\int 3(x+1)^2\,dx$

6 $\int 8(2x+1)^3\,dx$

7 $\int 6(2x+1)^2\,dx$

8 $\int 12(3x+3)^3\,dx$

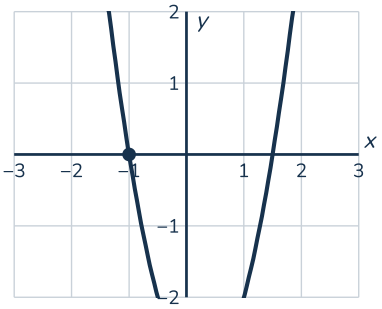
9 $\int -4(-x+2)^3\,dx$

10 $\int -6(-2x-2)^2\,dx$

6. Using derivatives

Find the equation of the tangent line at the point given.

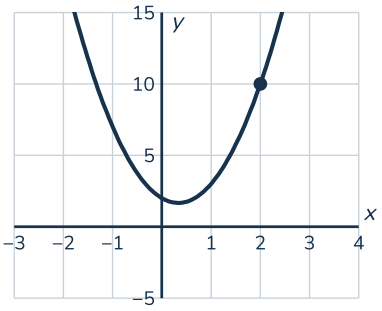
11 $f(x) = 2x^2 - x - 3,\ x = -1$



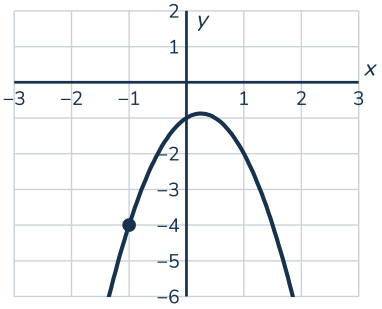
6. Using derivatives (continued)

Find the equation of the tangent line at the point given.

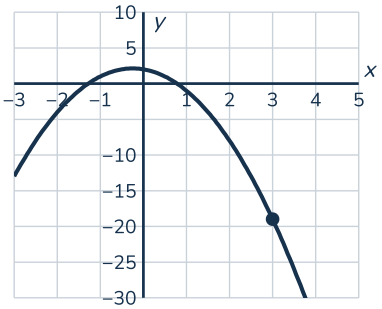
12 $f(x) = 3x^2 - 2x + 2,\ x = 2$



13 $f(x) = -2x^2 + x - 1,\ x = -1$



14 $f(x) = -2x^2 - x + 2,\ x = 3$



Integrals of sin, cos, e^x and 1/x

WORKED EXAMPLE

$$\int \left(-5 \sin x + \frac{1}{x}\right) dx$$
$$= 5 \cos x + \ln|x| + C$$

7. Integrals

Find each integral. Use the integral of each function, and divide by what the inside multiplies x by.

1 $\int \left(-4 \cos x - \frac{1}{x}\right) dx$

2 $\int \left(-\cos x - \sin x - \frac{1}{x}\right) dx$

3 $\int \left(\cos x - 2 \sin x - \frac{5}{x}\right) dx$

4 $\int (3 \sin x + 2e^x) \, dx$

5 $\int (-3 \sin x + 4e^x) \, dx$

6 $\int \left(-4 \sin x + 2e^x - \frac{5}{x}\right) dx$

7 $\int (-3 \cos x + e^x) \, dx$

8 $\int (\cos x + 2 \sin x + 2e^x) \, dx$

9 $\int \left(-2 \sin x - e^x + \frac{5}{x}\right) dx$

10 $\int \left(\cos x - \frac{1}{x}\right) dx$

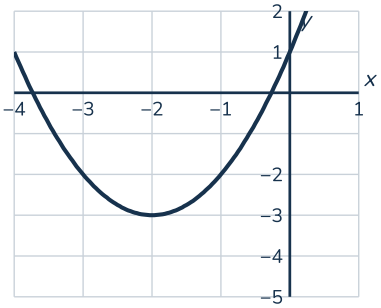
11 $\int (2 \sin x + 5e^x) \, dx$

12 $\int (-3 \cos x - 3 \sin x - 3e^x) \, dx$

8. Using derivatives

Find where f'(x) = 0 and say if each point is a maximum or a minimum.

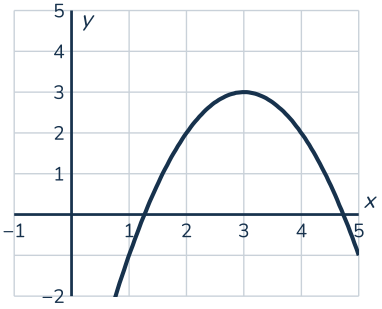
13 $f(x) = x^2 + 4x + 1$



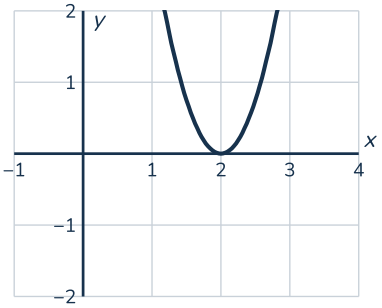
8. Using derivatives (continued)

Find where f'(x) = 0 and say if each point is a maximum or a minimum.

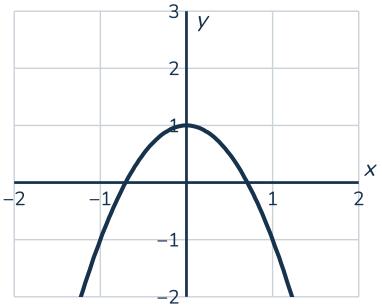
14 **$f(x) = -x^2 + 6x - 6$**



15 **$f(x) = 3x^2 - 12x + 12$**



16 **$f(x) = -2x^2 + 1$**



Integration by Parts

WORKED EXAMPLE

$\int -2x \sin x \, dx$

$u = -2x,\ dv = \sin x \, dx$

$du = -2 \, dx,\ v = -\cos x$

$= 2x \cos x - \int 2 \cos x \, dx$

$= 2x \cos x - 2 \sin x + C$

9. Integrals

Find each integral by parts: $\int u \, dv = uv - \int v \, du$.

1 $\int -4x \cos x \, dx$

2 $\int -x \cos x \, dx$

3 $\int 5x \sin x \, dx$

4 $\int -4xe^x \, dx$

5 $\int -5xe^x \, dx$

6 $\int 3x \cos x \, dx$

7 $\int -5x \cos x \, dx$

8 $\int -x \sin x \, dx$

10. Derivatives

Find f'(x) with the product rule: u'v + uv'.

9 $f(x) = (-2x + 3)(x + 3)$

10 $f(x) = (-x + 2)(2x + 1)$

11 $f(x) = (3x - 3)(-2x - 2)$

12 $f(x) = (-2x - 2)(-x - 3)$

13 $f(x) = (-2x - 2)(3x - 3)$

14 $f(x) = (2x + 2)(3x - 1)$

Answer Key (1 of 2)

Indefinite Integrals

Integrals

1. $\int(-4x)\,dx = -2x^2 + C$
2. $\int(2x - 1)\,dx = x^2 - x + C$
3. $\int(-4x + 2)\,dx = -2x^2 + 2x + C$
4. $\int(6x)\,dx = 3x^2 + C$
5. $\int(-4x - 1)\,dx = -2x^2 - x + C$
6. $\int(2x + 3)\,dx = x^2 + 3x + C$
7. $\int(-2x - 2)\,dx = -x^2 - 2x + C$
8. $\int(4x)\,dx = 2x^2 + C$
9. $\int(-6x - 2)\,dx = -3x^2 - 2x + C$
10. $\int(4x - 1)\,dx = 2x^2 - x + C$
11. $\int(-4x - 2)\,dx = -2x^2 - 2x + C$
12. $\int(-4x + 1)\,dx = -2x^2 + x + C$

Derivatives

13. $f(x) = -2x^2 - 3 \rightarrow f'(x) = -4x$
14. $f(x) = x^2 - x - 2 \rightarrow f'(x) = 2x - 1$
15. $f(x) = -3x^2 + x - 2 \rightarrow f'(x) = -6x + 1$
16. $f(x) = -x^2 - 2 \rightarrow f'(x) = -2x$
17. $f(x) = -2x^2 + 1 \rightarrow f'(x) = -4x$
18. $f(x) = -3x^2 + x + 2 \rightarrow f'(x) = -6x + 1$

Definite Integrals

Integrals

1. $\int$ from -1 to 0 of $(4x)\,dx = -2$
2. $\int$ from -1 to 0 of $(2x - 2)\,dx = -3$
3. $\int$ from 0 to 2 of $(4x)\,dx = 8$
4. $\int$ from 0 to 1 of $(4x)\,dx = 2$
5. $\int$ from -2 to 2 of $(4x - 2)\,dx = -8$
6. $\int$ from 0 to 2 of $(-4x - 2)\,dx = -12$
7. $\int$ from -1 to 1 of $(-4x + 2)\,dx = 4$
8. $\int$ from 0 to 1 of $(4x + 2)\,dx = 4$
9. $\int$ from -1 to 0 of $(4x + 2)\,dx = 0$
10. $\int$ from 0 to 2 of $(-2x)\,dx = -4$

Using derivatives

11. $f(x) = -x^2 + 1$; slope at $x = -2 \rightarrow f'(-2) = 4$
12. $f(x) = 2x^2 - 2$; slope at $x = 2 \rightarrow f'(2) = 8$
13. $f(x) = -2x^2 - 3x + 2$; slope at $x = -1 \rightarrow f'(-1) = 1$
14. $f(x) = 3x^2 + 3x - 2$; slope at $x = -1 \rightarrow f'(-1) = -3$

Integration by Substitution

Integrals

1. $\int 6(2x - 2)^2\,dx = (2x - 2)^3 + C$
2. $\int -9(-3x + 3)^2\,dx = (-3x + 3)^3 + C$
3. $\int 6(2x - 3)^2\,dx = (2x - 3)^3 + C$
4. $\int -12(-3x - 3)^3\,dx = (-3x - 3)^4 + C$
5. $\int 3(x + 1)^2\,dx = (x + 1)^3 + C$
6. $\int 8(2x + 1)^3\,dx = (2x + 1)^4 + C$

Integration by Substitution (continued)

Integrals (continued)

7. $\int 6(2x + 1)^2\,dx = (2x + 1)^3 + C$
8. $\int 12(3x + 3)^3\,dx = (3x + 3)^4 + C$
9. $\int -4(-x + 2)^3\,dx = (-x + 2)^4 + C$
10. $\int -6(-2x - 2)^2\,dx = (-2x - 2)^3 + C$

Using derivatives

11. $f(x) = 2x^2 - x - 3$; tangent at $x = -1 \rightarrow y = -5x - 5$
12. $f(x) = 3x^2 - 2x + 2$; tangent at $x = 2 \rightarrow y = 10x - 10$
13. $f(x) = -2x^2 + x - 1$; tangent at $x = -1 \rightarrow y = 5x + 1$
14. $f(x) = -2x^2 - x + 2$; tangent at $x = 3 \rightarrow y = -13x + 20$

Integrals of sin, cos, e^x and 1/x

Integrals

1. $\int (-4\cos x - 1/x)\,dx = -4\sin x - \ln|x| + C$
2. $\int (-\cos x - \sin x - 1/x)\,dx = -\sin x + \cos x - \ln|x| + C$
3. $\int (\cos x - 2\sin x - 5/x)\,dx = \sin x + 2\cos x - 5\ln|x| + C$
4. $\int (3\sin x + 2e^x)\,dx = -3\cos x + 2e^x + C$
5. $\int (-3\sin x + 4e^x)\,dx = 3\cos x + 4e^x + C$
6. $\int (-4\sin x + 2e^x - 5/x)\,dx = 4\cos x + 2e^x - 5\ln|x| + C$
7. $\int (-3\cos x + e^x)\,dx = -3\sin x + e^x + C$
8. $\int (\cos x + 2\sin x + 2e^x)\,dx = \sin x - 2\cos x + 2e^x + C$
9. $\int (-2\sin x - e^x + 5/x)\,dx = 2\cos x - e^x + 5\ln|x| + C$
10. $\int (\cos x - 1/x)\,dx = \sin x - \ln|x| + C$
11. $\int (2\sin x + 5e^x)\,dx = -2\cos x + 5e^x + C$
12. $\int (-3\cos x - 3\sin x - 3e^x)\,dx = -3\sin x + 3\cos x - 3e^x + C$

Using derivatives

13. $f(x) = x^2 + 4x + 1$; stationary points → minimum $(-2, -3)$
14. $f(x) = -x^2 + 6x - 6$; stationary points → maximum $(3, 3)$
15. $f(x) = 3x^2 - 12x + 12$; stationary points → minimum $(2, 0)$
16. $f(x) = -2x^2 + 1$; stationary points → maximum $(0, 1)$

Integration by Parts

Integrals

1. $\int -4x\cos x\,dx = -4x\sin x - 4\cos x + C$
2. $\int -x\cos x\,dx = -x\sin x - \cos x + C$
3. $\int 5x\sin x\,dx = -5x\cos x + 5\sin x + C$
4. $\int -4xe^x\,dx = -4xe^x + 4e^x + C$
5. $\int -5xe^x\,dx = -5xe^x + 5e^x + C$
6. $\int 3x\cos x\,dx = 3x\sin x + 3\cos x + C$
7. $\int -5x\cos x\,dx = -5x\sin x - 5\cos x + C$
8. $\int -x\sin x\,dx = x\cos x - \sin x + C$

Derivatives

9. $f(x) = (-2x + 3)(x + 3) \rightarrow f'(x) = -4x - 3$
10. $f(x) = (-x + 2)(2x + 1) \rightarrow f'(x) = -4x + 3$
11. $f(x) = (3x - 3)(-2x - 2) \rightarrow f'(x) = -12x$
12. $f(x) = (-2x - 2)(-x - 3) \rightarrow f'(x) = 4x + 8$
13. $f(x) = (-2x - 2)(3x - 3) \rightarrow f'(x) = -12x$

Answer Key (2 of 2)

Integration by Parts (continued)

Derivatives (continued)

14. $f(x) = (2x + 2)(3x - 1) \rightarrow f'(x) = 12x + 4$

Solutions (1 of 2)

Indefinite Integrals

Integrals

1. $\int (-4x)\,dx$

$$\int (-4x)\,dx = -\frac{4x^2}{2} + C$$

$$= -2x^2 + C$$

Integrate term by term, running the power rule backwards: add one to each power, then divide by the new power. A number on its own becomes that number times x. Add + C, because any constant would have disappeared when the answer was differentiated.

Divide out each fraction. Check by differentiating: you should get back the question.

2. $\int (2x - 1)\,dx$

$$\int (2x - 1)\,dx = \frac{2x^2}{2} - 1x + C$$

$$= x^2 - x + C$$

Integrate term by term, running the power rule backwards: add one to each power, then divide by the new power. A number on its own becomes that number times x. Add + C, because any constant would have disappeared when the answer was differentiated.

Divide out each fraction. Check by differentiating: you should get back the question.

Derivatives

13. $f(x) = -2x^2 - 3$

$$f'(x) = -2 \cdot 2x$$

$$= -4x$$

A sum of powers of x: take the derivative term by term. For each term, bring the power down in front as a multiplier and take one off the power — the derivative of x^n is nx^{n-1}. The constant −3 does not change as x moves, so its derivative is 0 and it is gone.

Multiply out each term. That is the derivative.

14. $f(x) = x^2 - x - 2$

$$f'(x) = 2x - 1$$

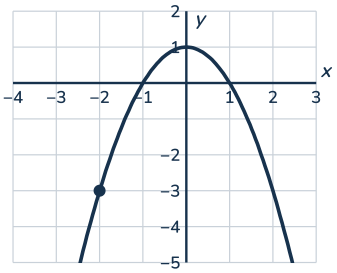
A sum of powers of x: take the derivative term by term. For each term, bring the power down in front as a multiplier and take one off the power — the derivative of x^n is nx^{n-1}. The constant −2 does not change as x moves, so its derivative is 0 and it is gone.

Solutions (2 of 2)

Definite Integrals

Using derivatives

11. **$f(x) = -x^2 + 1,\ x = -2$**



$$f'(x) = -2x$$

$$f'(-2) = -2(-2)$$

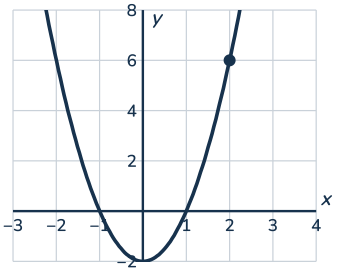
$$= 4$$

The slope of a curve at a point is the derivative there. Find f'(x) first, term by term.

Put x = −2 into f'(x), not into f(x): f(x) is the height, f'(x) is the slope.

Work it out. That is the slope of the curve at that point.

12. **$f(x) = 2x^2 - 2,\ x = 2$**



$$f'(x) = 4x$$

$$f'(2) = 4(2)$$

$$= 8$$

The slope of a curve at a point is the derivative there. Find f'(x) first, term by term.

Put x = 2 into f'(x), not into f(x): f(x) is the height, f'(x) is the slope.

Work it out. That is the slope of the curve at that point.