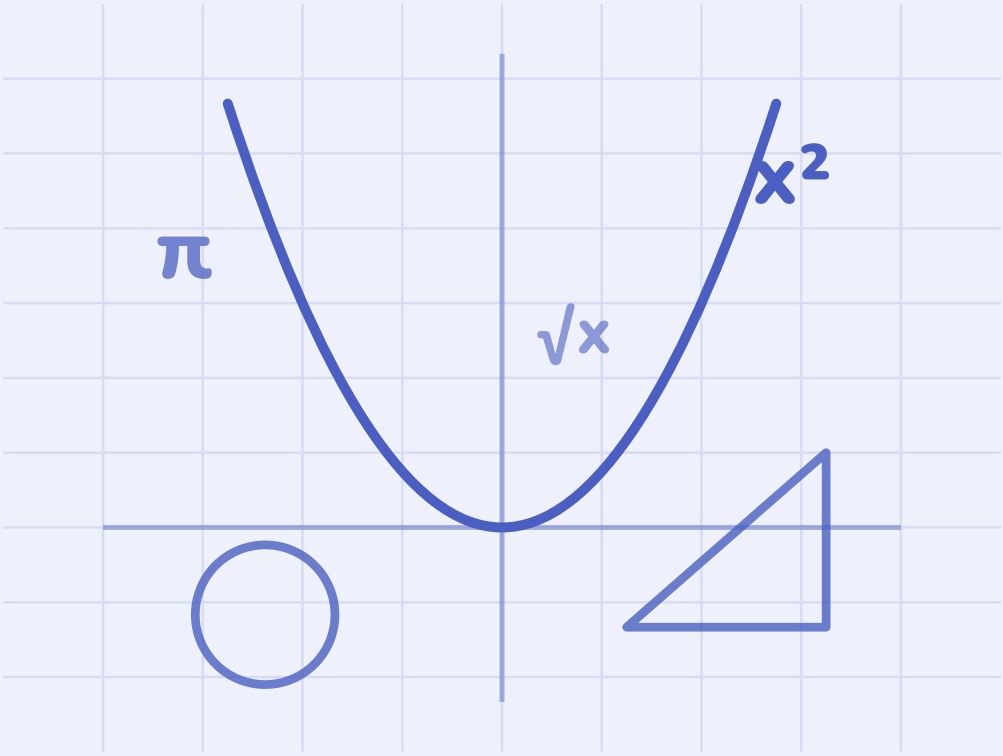
MATH PICNIC · WORKBOOK

Calculus Math Review Workbook

5 lessons · Ages 17–18 · answer key at the back



Name _____ Date _____

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1-1

Choosing the Rule

WORKED EXAMPLE

$f(x) = x^2 + x - 2$

$f'(x) = 2x + 1$

1. Derivatives

Find f'(x). Decide which rule the function needs before you start.

1 $f(x) = -2x^2 + 1$

2 $f(x) = (-2x + 1)^2$

3 $f(x) = (-x + 2)(-3x - 1)$

4 $f(x) = 2x^2 - 2x - 2$

5 $f(x) = (-2x - 3)^2$

6 $f(x) = (-2x + 1)(-x + 1)$

7 $f(x) = -3x^2 - 2x + 1$

8 $f(x) = (x + 3)^2$

9 $f(x) = \dfrac{2x + 1}{2x - 3}$

10 $f(x) = x^2 - 3$

2. Derivatives

Find f'(x) with the product rule: u'v + uv'.

11 $f(x) = (3x - 1)(x + 1)$

12 $f(x) = (-x + 1)(x - 1)$

13 $f(x) = (2x - 2)(-2x + 2)$

14 $f(x) = (-x - 1)(x - 1)$

15 $f(x) = (-x - 3)(-2x)$

16 $f(x) = (x - 3)(3x - 2)$

Product and Quotient Rules with Functions

WORKED EXAMPLE

$f(x) = 2e^x \cos x$

$u = 2e^x$, $v = \cos x$
$u' = 2e^x$, $v' = -\sin x$
$f'(x) = 2e^x \cos x - 2e^x \sin x$
$= 2e^x(\cos x - \sin x)$

3. Derivatives

Find f'(x). Decide whether f is a product, a quotient or a chain, then take u and v.

1 **$f(x) = 4e^x \cos x$**

2 **$f(x) = \dfrac{\sin x}{x}$**

3 **$f(x) = 2e^x \sin x$**

4 **$f(x) = 4e^x \sin x$**

5 **$f(x) = \dfrac{3e^x}{x}$**

6 **$f(x) = -4e^x \sin x$**

7 **$f(x) = \dfrac{\cos x}{x}$**

8 **$f(x) = 3e^x \cos x$**

4. Derivatives

Find f'(x). Decide which rule the function needs before you start.

9 **$f(x) = -3x^2 + 2x$**

10 **$f(x) = (x + 2)^3$**

11 **$f(x) = -2x^2 - 2x - 1$**

12 **$f(x) = (-x + 1)^2$**

4. Derivatives (continued)

Find f'(x). Decide which rule the function needs before you start.

13 $f(x) = (-2x + 3)(-3x + 3)$

14 $f(x) = (-2x - 1)^2$

Stationary Points

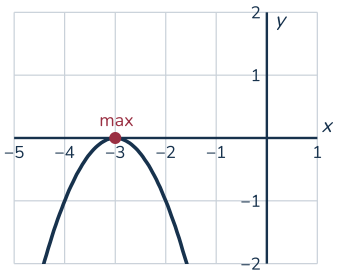
WORKED EXAMPLE

$f(x) = -x^2 - 6x - 9$

$f'(x) = -2x - 6 = 0$

$x = -3$, $f(-3) = 0$

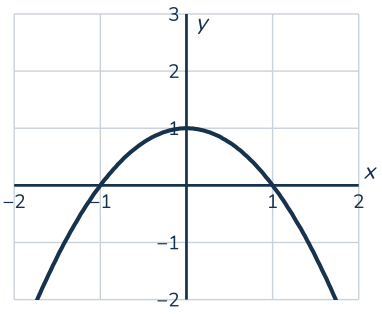
$f''(x) = -2 < 0$, so a maximum



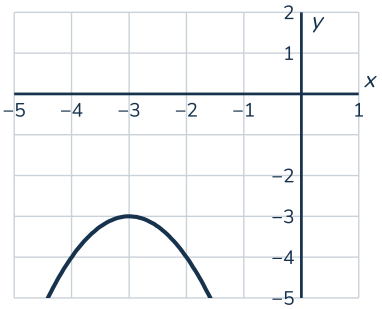
5. Using derivatives

Find where $f'(x) = 0$ and say if each point is a maximum or a minimum.

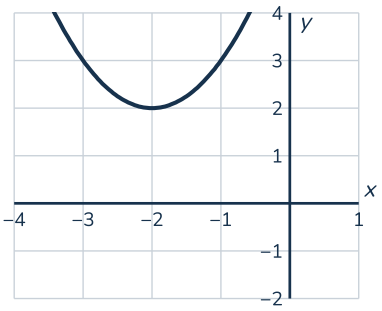
1 **$f(x) = -x^2 + 1$**



2 **$f(x) = -x^2 - 6x - 12$**



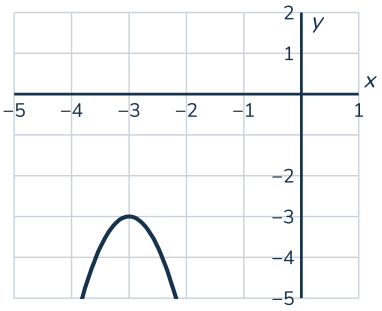
3 **$f(x) = x^2 + 4x + 6$**



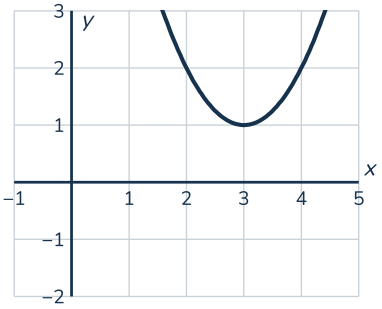
5. Using derivatives (continued)

Find where f'(x) = 0 and say if each point is a maximum or a minimum.

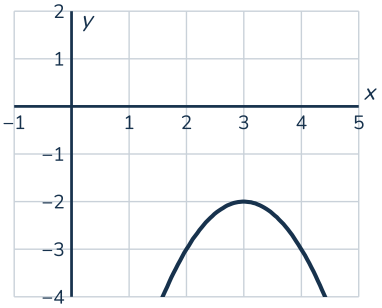
4 $f(x) = -3x^2 - 18x - 30$



5 $f(x) = x^2 - 6x + 10$



6 $f(x) = -x^2 + 6x - 11$



6. Derivatives

Find f'(x). Decide whether f is a product, a quotient or a chain, then take u and v.

7 $f(x) = -4e^x \sin x$

8 $f(x) = \dfrac{2 \ln x}{x}$

9 $f(x) = -e^x \cos x$

10 $f(x) = \dfrac{3 \cos x}{x}$

Integration by Parts

WORKED EXAMPLE

$\int -2x \sin x \, dx$

$u = -2x,\ dv = \sin x \, dx$

$du = -2 \, dx,\ v = -\cos x$

$= 2x \cos x - \int 2 \cos x \, dx$

$= 2x \cos x - 2 \sin x + C$

7. Integrals

Find each integral by parts: $\int u \, dv = uv - \int v \, du$.

1 $\int -x \cos x \, dx$

2 $\int -4x \cos x \, dx$

3 $\int 5x \sin x \, dx$

4 $\int -x \sin x \, dx$

5 $\int -4xe^x \, dx$

6 $\int -5xe^x \, dx$

7 $\int 2x \cos x \, dx$

8 $\int 3x \cos x \, dx$

8. Derivatives

Find f'(x) with the product rule: u'v + uv'.

9 $f(x) = (-x + 2)(2x + 1)$

10 $f(x) = (-2x + 3)(x + 3)$

11 $f(x) = (3x - 3)(-2x - 2)$

12 $f(x) = (-x + 2)(-3x - 1)$

13 $f(x) = (-2x - 2)(-x - 3)$

14 $f(x) = (-2x - 2)(3x - 3)$

Interval of Convergence

WORKED EXAMPLE

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n}$$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x+2|}{3}$$

$L = \frac{|x+2|}{3} < 1$: $R = 3$, $-5 < x < 1$

$x = 1$: $\sum 1$ diverges ($a_n \not\to 0$)

$x = -5$: $\sum (-1)^n$ diverges ($a_n \not\to 0$)

Interval: $(-5, 1)$

9. Series: converge or diverge?

Find the interval of convergence. Check both endpoints.

1 $$\sum_{n=1}^{\infty} \frac{(x+3)^n}{n \cdot 2^n}$$

2 $$\sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n}$$

3 $$\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{n \cdot 3^n}$$

4 $$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^2}$$

10. Derivatives

Find f'(x) with the product rule: u'v + uv'.

5 $f(x) = (3x + 2)(-3x + 1)$

6 $f(x) = (-x - 1)(-x + 2)$

7 $f(x) = (3x + 3)(x - 3)$

8 $f(x) = (x - 2)(3x - 2)$

Answer Key

Choosing the Rule

Derivatives

1. $f(x) = -2x^2 + 1 \rightarrow f'(x) = -4x$
2. $f(x) = (-2x + 1)^2 \rightarrow f'(x) = -4(-2x + 1)$
3. $f(x) = (-x + 2)(-3x - 1) \rightarrow f'(x) = 6x - 5$
4. $f(x) = 2x^2 - 2x - 2 \rightarrow f'(x) = 4x - 2$
5. $f(x) = (-2x - 3)^2 \rightarrow f'(x) = -4(-2x - 3)$
6. $f(x) = (-2x + 1)(-x + 1) \rightarrow f'(x) = 4x - 3$
7. $f(x) = -3x^2 - 2x + 1 \rightarrow f'(x) = -6x - 2$
8. $f(x) = (x + 3)^2 \rightarrow f'(x) = 2(x + 3)$
9. $f(x) = \frac{2x + 1}{2x - 3} \rightarrow f'(x) = -\frac{8}{2x - 3}^2$
10. $f(x) = x^2 - 3 \rightarrow f'(x) = 2x$

Derivatives

11. $f(x) = (3x - 1)(x + 1) \rightarrow f'(x) = 6x + 2$
12. $f(x) = (-x + 1)(x - 1) \rightarrow f'(x) = -2x + 2$
13. $f(x) = (2x - 2)(-2x + 2) \rightarrow f'(x) = -8x + 8$
14. $f(x) = (-x - 1)(x - 1) \rightarrow f'(x) = -2x$
15. $f(x) = (-x - 3)(-2x) \rightarrow f'(x) = 4x + 6$
16. $f(x) = (x - 3)(3x - 2) \rightarrow f'(x) = 6x - 11$

Product and Quotient Rules with Functions

Derivatives

1. $f(x) = 4e^x \cos x \rightarrow f'(x) = 4e^x(\cos x - \sin x)$
2. $f(x) = \frac{\sin x}{x} \rightarrow f'(x) = \frac{x \cos x - \sin x}{x^2}$
3. $f(x) = 2e^x \sin x \rightarrow f'(x) = 2e^x(\sin x + \cos x)$
4. $f(x) = 4e^x \sin x \rightarrow f'(x) = 4e^x(\sin x + \cos x)$
5. $f(x) = \frac{3e^x}{x} \rightarrow f'(x) = \frac{3e^x(x - 1)}{x^2}$
6. $f(x) = -4e^x \sin x \rightarrow f'(x) = -4e^x(\sin x + \cos x)$
7. $f(x) = \frac{\cos x}{x} \rightarrow f'(x) = \frac{-(x \sin x + \cos x)}{x^2}$
8. $f(x) = 3e^x \cos x \rightarrow f'(x) = 3e^x(\cos x - \sin x)$

Derivatives

9. $f(x) = -3x^2 + 2x \rightarrow f'(x) = -6x + 2$
10. $f(x) = (x + 2)^3 \rightarrow f'(x) = 3(x + 2)^2$
11. $f(x) = -2x^2 - 2x - 1 \rightarrow f'(x) = -4x - 2$
12. $f(x) = (-x + 1)^2 \rightarrow f'(x) = -2(-x + 1)$
13. $f(x) = (-2x + 3)(-3x + 3) \rightarrow f'(x) = 12x - 15$
14. $f(x) = (-2x - 1)^2 \rightarrow f'(x) = -4(-2x - 1)$

Stationary Points

Using derivatives

1. $f(x) = -x^2 + 1$; stationary points → maximum (0, 1)
2. $f(x) = -x^2 - 6x - 12$; stationary points → maximum (−3, −3)
3. $f(x) = x^2 + 4x + 6$; stationary points → minimum (−2, 2)
4. $f(x) = -3x^2 - 18x - 30$; stationary points → maximum (−3, −3)
5. $f(x) = x^2 - 6x + 10$; stationary points → minimum (3, 1)
6. $f(x) = -x^2 + 6x - 11$; stationary points → maximum (3, −2)

Stationary Points (continued)

Derivatives

7. $f(x) = -4e^x \sin x \rightarrow f'(x) = -4e^x(\sin x + \cos x)$
8. $f(x) = \frac{2 \ln x}{x} \rightarrow f'(x) = \frac{2(1 - \ln x)}{x^2}$
9. $f(x) = -e^x \cos x \rightarrow f'(x) = -e^x(\cos x - \sin x)$
10. $f(x) = \frac{3 \cos x}{x} \rightarrow f'(x) = \frac{-3(x \sin x + \cos x)}{x^2}$

Integration by Parts

Integrals

1. $\int -x \cos x \, dx = -x \sin x - \cos x + C$
2. $\int -4x \cos x \, dx = -4x \sin x - 4 \cos x + C$
3. $\int 5x \sin x \, dx = -5x \cos x + 5 \sin x + C$
4. $\int -x \sin x \, dx = x \cos x - \sin x + C$
5. $\int -4xe^x \, dx = -4xe^x + 4e^x + C$
6. $\int -5xe^x \, dx = -5xe^x + 5e^x + C$
7. $\int 2x \cos x \, dx = 2x \sin x + 2 \cos x + C$
8. $\int 3x \cos x \, dx = 3x \sin x + 3 \cos x + C$

Derivatives

9. $f(x) = (-x + 2)(2x + 1) \rightarrow f'(x) = -4x + 3$
10. $f(x) = (-2x + 3)(x + 3) \rightarrow f'(x) = -4x - 3$
11. $f(x) = (3x - 3)(-2x - 2) \rightarrow f'(x) = -12x$
12. $f(x) = (-x + 2)(-3x - 1) \rightarrow f'(x) = 6x - 5$
13. $f(x) = (-2x - 2)(-x - 3) \rightarrow f'(x) = 4x + 8$
14. $f(x) = (-2x - 2)(3x - 3) \rightarrow f'(x) = -12x$

Interval of Convergence

Series: converge or diverge?

1. $\sum_{n=1}^{\infty} \frac{(x + 3)^n}{n \cdot 2^n} \rightarrow [-5, -1)$
2. $\sum_{n=1}^{\infty} \frac{(x - 1)^n}{3^n} \rightarrow (-2, 4)$
3. $\sum_{n=1}^{\infty} \frac{(-1)^n(x + 1)^n}{n \cdot 3^n} \rightarrow (-4, 2]$
4. $\sum_{n=1}^{\infty} \frac{(x + 2)^n}{n^2} \rightarrow [-3, -1]$

Derivatives

5. $f(x) = (3x + 2)(-3x + 1) \rightarrow f'(x) = -18x - 3$
6. $f(x) = (-x - 1)(-x + 2) \rightarrow f'(x) = 2x - 1$
7. $f(x) = (3x + 3)(x - 3) \rightarrow f'(x) = 6x - 6$
8. $f(x) = (x - 2)(3x - 2) \rightarrow f'(x) = 6x - 8$

Solutions (1 of 4)

Choosing the Rule

Derivatives

1. **$f(x) = -2x^2 + 1$**

$$f'(x) = -2 \cdot 2x$$

$$\mathbf{= -4x}$$

A sum of powers of x: take the derivative term by term. For each term, bring the power down in front as a multiplier and take one off the power — the derivative of x^n is nx^{n-1}. The constant 1 does not change as x moves, so its derivative is 0 and it is gone.

Multiply out each term. That is the derivative.

2. **$f(x) = (-2x + 1)^2$**

$$u = -2x + 1,\ u' = -2$$

$$f'(x) = 2(-2x + 1) \cdot (-2)$$

$$\mathbf{= -4(-2x + 1)}$$

f is a bracket raised to a power, so the bracket is the inside function: $u = -2x + 1$. Its derivative is $u' = -2$.

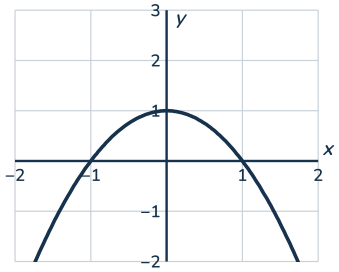
Differentiate the outside as if the bracket were x — bring the 2 down and take one off the power — and leave the bracket alone inside. Then multiply by u'.

Multiply the numbers in front. The bracket stays: that is the form the answer is left in.

Stationary Points

Using derivatives

1. **$f(x) = -x^2 + 1$**



$$f'(x) = -2x = 0$$

$$x = 0,\ f(0) = 1$$

$$\mathbf{f''(x) = -2 < 0,\ so\ a\ maximum}$$

A stationary point is where the slope is 0. Find f'(x) and set it equal to 0.

Solve for x, then put that x into f(x) to find the height of the point.

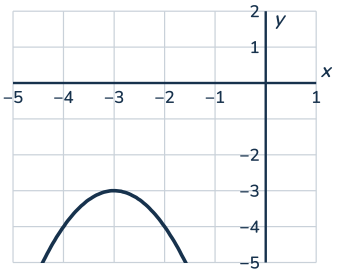
f'' says which kind: positive means the curve bends up, a minimum; negative means a maximum.

Solutions (2 of 4)

Stationary Points (continued)

Using derivatives (continued)

2. $f(x) = -x^2 - 6x - 12$



$$f'(x) = -2x - 6 = 0$$

$$x = -3,\ f(-3) = -3$$

$f''(x) = -2 < 0$, so a maximum

A stationary point is where the slope is 0. Find f'(x) and set it equal to 0.

Solve for x, then put that x into f(x) to find the height of the point.

f" says which kind: positive means the curve bends up, a minimum; negative means a maximum.

Integration by Parts

Integrals

1. $\int -x \cos x \, dx$

$$u = -x,\ dv = \cos x \, dx$$

$$du = -dx,\ v = \sin x$$

$$= -x \sin x + \int \sin x \, dx$$

$$= -x \sin x - \cos x + C$$

A product of x and a function: integrate by parts, $\int u \, dv = uv - \int v \, du$. Take $u = -x$, because it gets simpler when differentiated, and $dv = \cos x \, dx$.

Differentiate u to get du, and integrate dv to get v.

Put them into $uv - \int v \, du$. The integral left over has no x in front any more.

Integrate it and add + C. Check by differentiating with the product rule.

Solutions (3 of 4)

Integration by Parts (continued)

Integrals (continued)

2. $\int -4x \cos x \, dx$

$u = -4x,\ dv = \cos x\, dx$

$du = -4\, dx,\ v = \sin x$

$= -4x \sin x + \int 4 \sin x\, dx$

$= -4x \sin x - 4 \cos x + C$

A product of x and a function: integrate by parts, $\int u\, dv = uv - \int v\, du$. Take $u = -4x$, because it gets simpler when differentiated, and $dv = \cos x\, dx$.

Differentiate u to get du, and integrate dv to get v.

Put them into $uv - \int v\, du$. The integral left over has no x in front any more.

Integrate it and add + C. Check by differentiating with the product rule.

Interval of Convergence

Series: converge or diverge?

1. $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n \cdot 2^n}$

$\frac{|a_{n+1}|}{|a_n|} = \frac{|x+3|}{2} \cdot \frac{n}{n+1}$

$L = \frac{|x+3|}{2}$

< 1: $R = 2$, $-5 < x < -1$

$x = -1$: $\sum \frac{1}{n}$

diverges ($p = 1$)

$x = -5$: $\sum (-1)^n \frac{1}{n}$

converges (alternating)

Interval: [−5, −1)

Use the ratio test, with x kept as a letter. Divide each term by the one before, size only.

As n grows, n over n + 1 goes to 1, and L is |x + 3| over 2. The series converges when $L < 1$, which is $|x + 3| < 2$: a radius of 2 around −3. Inside it the series converges and outside it diverges, but the ratio test says nothing AT the ends, so each one is checked on its own.

At $x = -1$, $x + 3 = 2$, so $(x + 3)^n$ is 2^n, which cancels the 2^n on the bottom. What is left is the harmonic series, which diverges.

At $x = -5$, $x + 3 = -2$, so $(x + 3)^n$ is $(-1)^n$ times 2^n, which cancels the 2^n on the bottom. What is left alternates, and its sizes shrink to 0, so it converges by the alternating series test.

So the series converges at the left end only. A square bracket includes an end, a round one leaves it out.

Solutions (4 of 4)

Interval of Convergence (continued)

Series: converge or diverge? (continued)

2. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n}$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x-1|}{3}$$

$$L = \frac{|x-1|}{3}$$

< 1: $R = 3$, $-2 < x < 4$

$x = 4$: $\sum 1$ diverges ($a_n \not\to 0$)

$x = -2$: $\sum (-1)^n$

diverges ($a_n \not\to 0$)

Interval: (−2, 4)

Use the ratio test, with x kept as a letter. Divide each term by the one before, size only.

As n grows, L is |x − 1| over 3. The series converges when L < 1, which is |x − 1| < 3: a radius of 3 around 1. Inside it the series converges and outside it diverges, but the ratio test says nothing AT the ends, so each one is checked on its own.

At x = 4, x − 1 = 3, so $(x-1)^n$ is 3^n, which cancels the 3^n on the bottom. What is left has terms that never go to 0, so it diverges.

At x = −2, x − 1 = −3, so $(x-1)^n$ is $(-1)^n$ times 3^n, which cancels the 3^n on the bottom. What is left has terms that never go to 0, so it diverges.

So the series converges at neither end. A square bracket includes an end, a round one leaves it out.