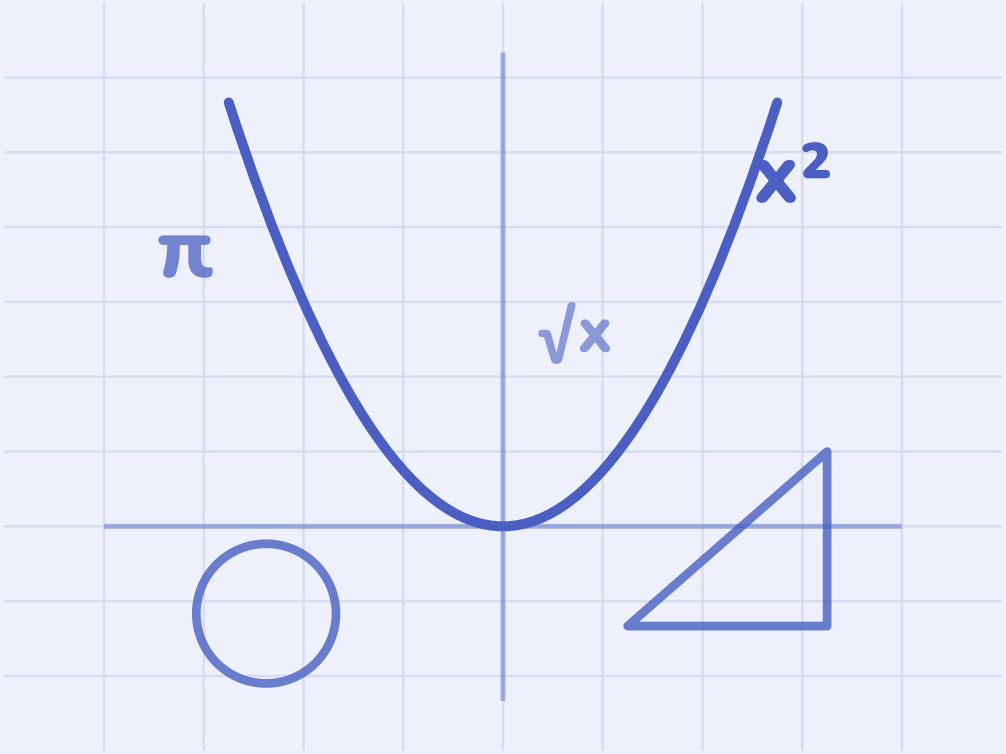
MATH PICNIC · WORKBOOK

Calculus Series Workbook

6 lessons · Ages 17–18 · answer key at the back



Name _____ Date _____

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1-1

The nth-Term Test

WORKED EXAMPLE

$$\sum_{n=1}^{\infty} \frac{2}{n^2 - n + 3}$$

$$\lim_{n\to\infty} \frac{2}{n^2 - n + 3} = \lim_{n\to\infty} \frac{2/n^2}{1 - 1/n + 3/n^2}$$

$$= \frac{0}{1 + 0 + 0} = 0$$

$\lim a_n = 0$: the test says nothing

1. Series: converge or diverge?

Find $\lim a_n$. Not 0: the series diverges. 0: the test says nothing.

1 $$\sum_{n=1}^{\infty} \frac{1}{n^2 + n + 3}$$

2 $$\sum_{n=1}^{\infty} \frac{-2n - 2}{2n^2 + 3}$$

3 $$\sum_{n=1}^{\infty} \frac{-n - 3}{n^2 + 1}$$

4 $$\sum_{n=1}^{\infty} \frac{-n^2 - 3}{2n^2 + 1}$$

5 $$\sum_{n=1}^{\infty} \frac{-2n^2 + 1}{n^2 + 2}$$

6 $$\sum_{n=1}^{\infty} \frac{2n - 2}{n^2 + 1}$$

7 $$\sum_{n=1}^{\infty} \frac{2n - 2}{2n + 3}$$

8 $$\sum_{n=1}^{\infty} \frac{2}{2n^2 + n - 1}$$

2. Integrals

Find the antiderivative, then take its value at the bottom from the top.

9 $\int_{1}^{2} (4x - 1)\,dx$

10 $\int_{0}^{1} (2x - 2)\,dx$

11 $\int_{-1}^{1} (2x)\,dx$

12 $\int_{0}^{2} (-2x + 2)\,dx$

Geometric and Telescoping Series

WORKED EXAMPLE

$$\sum_{n=1}^{\infty} 3(-4)^{n-1}$$

$a = 3$ and $r = -4$

$|r| = 4 > 1$, so the series diverges

3. Series: converge or diverge?

Does each series converge? If it does, find its sum.

1 $$\sum_{n=1}^{\infty} -2(-4)^{n-1}$$

2 $$\sum_{n=1}^{\infty} -2\left(-\frac{1}{4}\right)^{n-1}$$

3 $$\sum_{n=0}^{\infty} 2\left(\frac{2}{3}\right)^{n}$$

4 $$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n}$$

5 $$\sum_{n=1}^{\infty} \frac{2}{(n+2)(n+3)}$$

6 $$\sum_{n=0}^{\infty} -2\left(\frac{2}{3}\right)^{n}$$

7 $$\sum_{n=1}^{\infty} 3(-2)^{n}$$

8 $$\sum_{n=0}^{\infty} 2 \cdot 4^{n}$$

4. Integrals

Find each integral by substitution: let u be the bracket.

9 $\int -6(-2x + 2)^2 \, dx$

10 $\int -8(-2x + 1)^3 \, dx$

11 $\int -9(-3x + 2)^2 \, dx$

12 $\int 3(x + 3)^2 \, dx$

p-Series and the Comparison Test

WORKED EXAMPLE

$$\sum_{n=1}^{\infty} \frac{3}{n}$$

A number times a p-series: $\frac{3}{n} = 3 \cdot \frac{1}{n}$

$p = 1 \le 1$, so the series diverges

5. Series: converge or diverge?

Does each series converge? Compare it with a p-series, $\sum 1/n^p$.

1 $$\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$$

2 $$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

3 $$\sum_{n=1}^{\infty} \frac{2}{n}$$

4 $$\sum_{n=1}^{\infty} \frac{n + 1}{2n^3 + 1}$$

5 $$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$$

6 $$\sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{n}}$$

7 $$\sum_{n=1}^{\infty} \frac{1}{n}$$

8 $$\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}}$$

6. Integrals

Find each integral. Use the integral of each function, and divide by what the inside multiplies x by.

9 $\int (-2 \sin x + e^x)\, dx$

10 $\int (5 \sin x + \frac{2}{x})\, dx$

11 $\int (3 \cos x - \sin x)\, dx$

12 $\int (3 \sin x - 3e^x + \frac{4}{x})\, dx$

The Ratio Test

WORKED EXAMPLE

$$\sum_{n=1}^{\infty} \frac{4^n}{n^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}}{(n+1)^2} \cdot \frac{n^2}{4^n}$$

$$= 4 \cdot \frac{n^2}{(n+1)^2}$$

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 4 \cdot 1 = 4$$

$L > 1$, so the series diverges

7. Series: converge or diverge?

Does each series converge? Use the ratio test: $L = \lim |a_{n+1} \div a_n|$.

1 $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

2 $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2}$

3 $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$

4 $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n!}$

5 $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$

6 $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n}$

7 $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{4^n}$

8 $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

8. Derivatives

Find f'(x). Use the derivative of each function, and the chain rule when there is an inside.

9 $f(x) = -3\cos x - \ln x$

10 $f(x) = 5\sin x - 5e^x$

11 $f(x) = 2e^x - 3\ln x$

12 $f(x) = -5\cos x + 3e^x - \ln x$

13 $f(x) = -5\sin x - 5\cos x$

14 $f(x) = -\cos x - 2e^x$

Alternating Series

WORKED EXAMPLE

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n\sqrt{n}}$$

$\sum |a_n| = \sum \frac{1}{n\sqrt{n}}$: $p = \frac{3}{2} > 1$, converges

The series converges absolutely

9. Series: converge or diverge?

Does each series converge absolutely, converge conditionally, or diverge?

1 $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{n+1}$$

2 $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

3 $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n^2\sqrt{n}}$$

4 $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}$$

5 $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2\sqrt{n}}$$

6 $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n}$$

7 $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{\sqrt{n}}$$

8 $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{2n+1}$$

10. Integrals

Raise each power by one and divide by the new power. Don't forget + C.

9 $\int (-2x + 1)\,dx$

10 $\int (-6x + 1)\,dx$

11 $\int (4x - 3)\,dx$

12 $\int (2x - 1)\,dx$

13 $\int (6x - 1)\,dx$

14 $\int (-6x + 2)\,dx$

Interval of Convergence

WORKED EXAMPLE

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n}$$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x+2|}{3}$$

$L = \frac{|x+2|}{3} < 1$: $R = 3$, $-5 < x < 1$

$x = 1$: $\sum 1$ diverges ($a_n \not\to 0$)

$x = -5$: $\sum (-1)^n$ diverges ($a_n \not\to 0$)

Interval: $(-5, 1)$

11. Series: converge or diverge?

Find the interval of convergence. Check both endpoints.

1 $$\sum_{n=1}^{\infty} \frac{(x+3)^n}{n \cdot 2^n}$$

2 $$\sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n}$$

3 $$\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{n \cdot 3^n}$$

4 $$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^2}$$

12. Derivatives

Find f'(x) with the product rule: u'v + uv'.

5 $f(x) = (3x + 2)(-3x + 1)$

6 $f(x) = (-x - 1)(-x + 2)$

7 $f(x) = (3x + 3)(x - 3)$

8 $f(x) = (x - 2)(3x - 2)$

Answer Key (1 of 2)

The nth-Term Test

Series: converge or diverge?

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+n+3}$ → the test says nothing
2. $\sum_{n=1}^{\infty} \frac{-2n-2}{2n^2+3}$ → the test says nothing
3. $\sum_{n=1}^{\infty} \frac{-n-3}{n^2+1}$ → the test says nothing
4. $\sum_{n=1}^{\infty} \frac{-n^2-3}{2n^2+1}$ → diverges
5. $\sum_{n=1}^{\infty} \frac{-2n^2+1}{n^2+2}$ → diverges
6. $\sum_{n=1}^{\infty} \frac{2n-2}{n^2+1}$ → the test says nothing
7. $\sum_{n=1}^{\infty} \frac{2n-2}{2n+3}$ → diverges
8. $\sum_{n=1}^{\infty} \frac{2}{2n^2+n-1}$ → the test says nothing

Integrals

9. $\int$ from 1 to 2 of $(4x-1)\,dx = 5$
10. $\int$ from 0 to 1 of $(2x-2)\,dx = -1$
11. $\int$ from -1 to 1 of $(2x)\,dx = 0$
12. $\int$ from 0 to 2 of $(-2x+2)\,dx = 0$

Geometric and Telescoping Series

Series: converge or diverge?

1. $\sum_{n=1}^{\infty} -2(-4)^{n-1}$ → diverges
2. $\sum_{n=1}^{\infty} -2(-\frac{1}{4})^{n-1}$ → converges, $S = \frac{-8}{5}$
3. $\sum_{n=0}^{\infty} 2(\frac{2}{3})^n$ → converges, $S = 6$
4. $\sum_{n=0}^{\infty} (\frac{2}{3})^n$ → converges, $S = 3$
5. $\sum_{n=1}^{\infty} \frac{2}{(n+2)(n+3)}$ → converges, $S = \frac{2}{3}$
6. $\sum_{n=0}^{\infty} -2(\frac{2}{3})^n$ → converges, $S = -6$
7. $\sum_{n=1}^{\infty} 3(-2)^n$ → diverges
8. $\sum_{n=0}^{\infty} 2 \cdot 4^n$ → diverges

Integrals

9. $\int -6(-2x+2)^2\,dx = (-2x+2)^3 + C$
10. $\int -8(-2x+1)^3\,dx = (-2x+1)^4 + C$
11. $\int -9(-3x+2)^2\,dx = (-3x+2)^3 + C$
12. $\int 3(x+3)^2\,dx = (x+3)^3 + C$

p-Series and the Comparison Test

Series: converge or diverge?

1. $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$ → converges
2. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ → converges
3. $\sum_{n=1}^{\infty} \frac{2}{n}$ → diverges
4. $\sum_{n=1}^{\infty} \frac{n+1}{2n^3+1}$ → converges
5. $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$ → diverges
6. $\sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{n}}$ → converges
7. $\sum_{n=1}^{\infty} \frac{1}{n}$ → diverges
8. $\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}}$ → converges

Integrals

9. $\int (-2\sin x + e^x)\,dx = 2\cos x + e^x + C$
10. $\int (5\sin x + 2/x)\,dx = -5\cos x + 2\ln|x| + C$
11. $\int (3\cos x - \sin x)\,dx = 3\sin x + \cos x + C$
12. $\int (3\sin x - 3e^x + 4/x)\,dx = -3\cos x - 3e^x + 4\ln|x| + C$

The Ratio Test

Series: converge or diverge?

1. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ → converges
2. $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$ → diverges
3. $\sum_{n=1}^{\infty} \frac{n}{3^n}$ → converges
4. $\sum_{n=1}^{\infty} \frac{4^n}{n!}$ → converges
5. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ → converges
6. $\sum_{n=1}^{\infty} \frac{3^n}{n}$ → diverges
7. $\sum_{n=1}^{\infty} \frac{n^2}{4^n}$ → converges
8. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ → diverges

Derivatives

9. $f(x) = -3\cos x - \ln x \rightarrow f'(x) = 3\sin x - \frac{1}{x}$
10. $f(x) = 5\sin x - 5e^x \rightarrow f'(x) = 5\cos x - 5e^x$
11. $f(x) = 2e^x - 3\ln x \rightarrow f'(x) = 2e^x - \frac{3}{x}$
12. $f(x) = -5\cos x + 3e^x - \ln x \rightarrow f'(x) = 5\sin x + 3e^x - \frac{1}{x}$
13. $f(x) = -5\sin x - 5\cos x \rightarrow f'(x) = -5\cos x + 5\sin x$
14. $f(x) = -\cos x - 2e^x \rightarrow f'(x) = \sin x - 2e^x$

Answer Key (2 of 2)

Alternating Series

Series: converge or diverge?

1. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{n+1}$ → diverges
2. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ → converges conditionally
3. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n^2\sqrt{n}}$ → converges absolutely
4. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}$ → diverges
5. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2\sqrt{n}}$ → converges absolutely
6. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n}$ → converges conditionally
7. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{\sqrt{n}}$ → converges conditionally
8. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{2n+1}$ → diverges

Integrals

9. $\int(-2x + 1)\,dx = -x^2 + x + C$
10. $\int(-6x + 1)\,dx = -3x^2 + x + C$
11. $\int(4x - 3)\,dx = 2x^2 - 3x + C$
12. $\int(2x - 1)\,dx = x^2 - x + C$
13. $\int(6x - 1)\,dx = 3x^2 - x + C$
14. $\int(-6x + 2)\,dx = -3x^2 + 2x + C$

Interval of Convergence

Series: converge or diverge?

1. $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n \cdot 2^n}$ → $[-5, -1)$
2. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n}$ → $(-2, 4)$
3. $\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{n \cdot 3^n}$ → $(-4, 2]$
4. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^2}$ → $[-3, -1]$

Derivatives

5. $f(x) = (3x + 2)(-3x + 1)$ → $f'(x) = -18x - 3$
6. $f(x) = (-x - 1)(-x + 2)$ → $f'(x) = 2x - 1$
7. $f(x) = (3x + 3)(x - 3)$ → $f'(x) = 6x - 6$
8. $f(x) = (x - 2)(3x - 2)$ → $f'(x) = 6x - 8$

Solutions (1 of 3)

The nth-Term Test

Series: converge or diverge?

1. $\sum_{n=1}^{\infty} \frac{1}{n^2 + n + 3}$

$$\lim_{n\to\infty} \frac{1}{n^2 + n + 3} = \lim_{n\to\infty} \frac{1/n^2}{1 + 1/n + 3/n^2}$$

$$= \frac{0}{1 + 0 + 0} = 0$$

$\lim a_n = 0$: the test says nothing

The nth-term test asks one thing: do the terms go to 0? Find the limit of a_n as n grows. The highest power on the bottom is n^2, so divide every term, top and bottom, by n^2.

Every term with n left on its bottom shrinks to 0, which leaves 0 on the top: the limit is 0.

The terms do go to 0, and that is NOT enough to say the series converges: the series of 1 over n has terms that go to 0 and still diverges. The test says nothing here, and another test is needed.

2. $\sum_{n=1}^{\infty} \frac{-2n - 2}{2n^2 + 3}$

$$\lim_{n\to\infty} \frac{-2n - 2}{2n^2 + 3} = \lim_{n\to\infty} \frac{-2/n - 2/n^2}{2 + 3/n^2}$$

$$= \frac{0 + 0}{2 + 0} = 0$$

$\lim a_n = 0$: the test says nothing

The nth-term test asks one thing: do the terms go to 0? Find the limit of a_n as n grows. The highest power on the bottom is n^2, so divide every term, top and bottom, by n^2.

Every term with n left on its bottom shrinks to 0, which leaves 0 on the top: the limit is 0.

The terms do go to 0, and that is NOT enough to say the series converges: the series of 1 over n has terms that go to 0 and still diverges. The test says nothing here, and another test is needed.

Solutions (2 of 3)

The nth-Term Test (continued)

Integrals

9. $\int_1^2 (4x - 1)\,dx$

$$= [2x^2 - x]_1^2$$

$$= (2(2)^2 - 2) - (2(1)^2 - 1)$$

$$= 6 - 1$$

$$= 5$$

Find an antiderivative first, the same way as an indefinite integral. The + C can be left out here: it would be added at the top and taken away at the bottom. Write it in square brackets with the limits 1 and 2 beside it.

Put the top limit in, then take away the value at the bottom limit: F(2) – F(1).

Work out each bracket: F(2) = 6 and F(1) = 1.

Subtract. That number is the integral.

10. $\int_0^1 (2x - 2)\,dx$

$$= [x^2 - 2x]_0^1$$

$$= ((1)^2 - 2(1)) - 0$$

$$= -1 - 0$$

$$= -1$$

Find an antiderivative first, the same way as an indefinite integral. The + C can be left out here: it would be added at the top and taken away at the bottom. Write it in square brackets with the limits 0 and 1 beside it.

Put the top limit in, then take away the value at the bottom limit: F(1) – F(0).

Work out each bracket: F(1) = –1 and F(0) = 0.

Subtract. That number is the integral.

The Ratio Test

Derivatives

9. $f(x) = -3\cos x - \ln x$

$$f'(x) = 3\sin x - \frac{1}{x}$$

A sum of functions: take the derivative of each term on its own, keeping its number in front. The four to know: the derivative of sin x is cos x, of cos x is –sin x, e^x is its own derivative, and the derivative of ln x is 1/x.

Solutions (3 of 3)

The Ratio Test (continued)

Derivatives (continued)

10. $f(x) = 5 \sin x - 5e^x$

$f'(x) = 5 \cos x - 5e^x$

A sum of functions: take the derivative of each term on its own, keeping its number in front. The four to know: the derivative of sin x is cos x, of cos x is −sin x, e^x is its own derivative, and the derivative of ln x is 1/x.